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Non-abelian finite gauge theories

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Abstract: We study orbifolds of $\mathcal{N} = 4$ $U(n)$ super-Yang-Mills theory given by discrete subgroups of $SU(2)$ and $SU(3)$. We have reached many interesting observations that have graph-theoretic interpretations. For the subgroups of $SU(2)$, we have shown how the matter content agrees with current quiver theories and have offered a possible explanation. In the case of $SU(3)$ we have constructed a catalogue of candidates for finite (chiral) $\mathcal{N} = 1$ theories, giving the gauge group and matter content. Finally, we conjecture a McKay-type correspondence for Gorenstein singularities in dimension 3 with modular invariants of WZW conformal models. This implies a connection between a class of finite $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions and the classification of affine $SU(3)$ modular invariant partition functions in two dimensions.

Keywords: Brane Dynamics in Gauge Theories, Conformal Field Models in String Theory.

Contents

1. Introduction

Recent advances on finite four dimensional gauge theories from string theory constructions have been dichotomous: either from the geometrical perspective of studying algebro-geometric singularities such as orbifolds [4, 5, 6], or from the intuitive perspective of studying various configurations of branes such as the so-called brane-box models [7]. (See [8] and references therein for a detailed description of these models. A recent paper discusses the bending of non-finite models in this context [9].) The two approaches lead to the realisation of finite, possibly chiral, $\mathcal{N} = 1$ supersymmetric gauge theories, such as those discussed in [10]. Our ultimate dream is of course to have the flexbility of the equivalence and completion of these approaches, allowing us to compute say, the duality group acting on the moduli space of marginal gauge couplings [11]. (The duality groups for the $\mathcal{N} = 2$ supersymmetric theories were discussed in the context of these two approaches in [12] and [13].) The brane-box method has met great success in providing the intuitive picture for orbifolds by

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Abelian groups: the elliptic model consisting of $k \times k'$ branes conveniently reproduces the theories on orbifolds by $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ [8]. Orbifolds by $\mathbb{Z}_k$ subgroups of $SU(3)$ are given by Brane Box Models with non-trivial identification on the torus [11, 8]. Since by the structure theorem that all finite Abelian groups are direct sums of cyclic ones, this procedure can be presumably extended to all Abelian quotient singularities. The non-Abelian groups however, present difficulties. By adding orientifold planes, the dihedral groups have also been successfully attacked for theories with $\mathcal{N}=2$ supersymmetry [14]. The question still remains as to what could be done for the myriad of finite groups, and thus to general Gorenstein singularities.

In this paper we shall present a catalogue of these Gorenstein singularities in dimensions 2 and 3, i.e., orbifolds constructed from discrete subgroups of $SU(2)$ and $SU(3)$ whose classification are complete. In particular we shall concentrate on the gauge group, the fermionic and bosonic matter content resulting from the orbifolding of an $\mathcal{N}=4$ $U(n)$ super-Yang-Mills theory. In Section 2, we present the general arguments that dictate the matter content for arbitrary finite group Γ. Then in Section 3, we study the case of $\Gamma \subset SU(2)$ where we notice interesting graph-theoretic descriptions of the matter matrices. We analogously analyse case by case, the discrete subgroups of $SU(3)$ in Section 4, followed by a brief digression of possible mathematical interest in Section 5. This leads to a Mckay-type connection between the classification of two dimensional $SU(3)_k$ modular invariant partition functions and the class of finite $\mathcal{N}=1$ supersymmetric gauge theories calculated in this paper. Finally we tabulate possible chiral theories obtainable by such orbifolding techniques for these $SU(3)$ subgroups.

2. The orbifolding technique

Prompted by works by Douglas, Greene, Moore and Morrison on gauge theories which arise by placing D3 branes on orbifold singularities [1, 2, 3], Kachru and Silverstein [4] and subsequently Lawrence, Nekrasov and Vafa [5] noted that an orbifold theory involving the projection of a supersymmetric $\mathcal{N}=4$ gauge theory on some discrete subgroup $\Gamma \subset SU(4)$ leads to a conformal field theory with $\mathcal{N} \leq 4$ supersymmetry. We shall first briefly summarise their results here.

We begin with a $U(n)$ $\mathcal{N}=4$ super-Yang-Mills theory which has an R-symmetry of $Spin(6) \simeq SU(4)$. There are gauge bosons A_{IJ} $(I,J=1,\ldots,n)$ being singlets of $Spin(6)$, along with adjoint Weyl fermions $\Psi^{\mathbf{4}}_{IJ}$ in the fundamental $\mathbf{4}$ of $SU(4)$ and adjoint scalars $\Phi^{\mathbf{6}}_{IJ}$ in the antisymmetric $\mathbf{6}$ of $SU(4)$. Then we choose a discrete (finite) subgroup $\Gamma \subset SU(4)$ with the set of irreducible representations $\{\mathbf{r}_i\}$ acting on the gauge group by breaking the I-indices up according to $\{\mathbf{r}_i\}$, i.e., by $\bigoplus\limits_i \mathbf{r}_i = \bigoplus\limits_i \mathbb{C}^{N_i}\mathbf{r}_i$ such that $\mathbb{C}^{N_i}$ accounts for the multiplicity of each $\mathbf{r}_i$ and $n = \sum\limits_{i=1} N_i \dim(\mathbf{r}_i)$. In the string theory picture, this decomposition of the gauge group corresponds to permuting n D3-branes and hence their Chan-Paton factors which contain the IJ indices, on orbifolds of $\mathbb{R}^6$. Subsequently by the Maldecena large N conjecture [15], we have an orbifold theory on $AdS_5 \times S^5$, with the R-symmetry manifesting as the $SO(6)$ symmetry group of S^5 in which the branes now live [4]. The string perturbative calculation in this context, especially with respect to vanishing theorems for β-functions, has been performed [6].

Having decomposed the gauge group, we must likewise do so for the matter fields: since an orbifold is invariant under the Γ-action, we perform the so-called projection on the fields by keeping only the Γ-invariant fields in the theory. Subsequently we arrive at a (superconformal) field theory with gauge group $G = \bigotimes\limits_i SU(N_i)$ and Yukawa and quartic interaction respectively as (in the notation of [5]):

$$\begin{aligned} Y &= \textstyle\sum_{ijk} \gamma_{ijk}^{f_{ij},f_{jk},f_{ki}} \mathrm{Tr}\Psi^{ij}_{f_{ij}}\Phi^{jk}_{f_{jk}}\Psi^{ki}_{f_{ki}} \\ V &= \textstyle\sum_{ijkl} \eta^{ijkl}_{f_{ij},f_{jk},f_{kl},f_{li}} \mathrm{Tr}\Phi^{ij}_{f_{ij}}\Phi^{jk}_{f_{jk}}\Phi^{kl}_{f_{kl}}\Phi^{li}_{f_{li}}, \end{aligned}$$

where

$$\begin{aligned} \gamma_{ijk}^{f_{ij},f_{jk},f_{ki}} &= \Gamma_{\alpha\beta,m}\left(Y_{f_{ij}}\right)^{\alpha}_{v_i\bar{v}_j}\left(Y_{f_{jk}}\right)^{m}_{v_j\bar{v}_k}\left(Y_{f_{ki}}\right)^{\beta}_{v_k\bar{v}_i} \\ \eta^{ijkl}_{f_{ij},f_{jk},f_{kl},f_{li}} &= \left(Y_{f_{ij}}\right)^{[m}_{v_i\bar{v}_j}\left(Y_{f_{jk}}\right)^{n]}_{v_j\bar{v}_k}\left(Y_{f_{kl}}\right)^{[m}_{v_k\bar{v}_l}\left(Y_{f_{li}}\right)^{n]}_{v_l\bar{v}_i}, \end{aligned}$$

such that $\left(Y_{f_{ij}}\right)^{\alpha}_{v_i\bar{v}_j}$, $\left(Y_{f_{ij}}\right)^{m}_{v_i\bar{v}_j}$ are the f_{ij}'th Clebsch-Gordan coefficients corresponding to the projection of $4 \otimes \mathbf{r}_i$ and $6 \otimes \mathbf{r}_i$ onto $\mathbf{r}_j$, and $\Gamma_{\alpha\beta,m}$ is the invariant in $\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{6}$.

Furthermore, the matter content is as follows:

1. Gauge bosons transforming as

$$\hom\left(\mathbb{C}^n,\mathbb{C}^n\right)^{\Gamma} = \bigoplus_i \mathbb{C}^{N_i}\otimes\left(\mathbb{C}^{N_i}\right)^*,$$

   which simply means that the original (R-singlet) adjoint $U(n)$ fields now break up according to the action of Γ to become the adjoints of the various $SU(N_i)$;

2. $a^{\mathbf{4}}_{ij}$ Weyl fermions $\Psi^{ij}_{f_{ij}}$ $(f_{ij} = 1,\ldots,a^{\mathbf{4}}_{ij})$

$$\left(\mathbf{4}\otimes\hom\left(\mathbb{C}^n,\mathbb{C}^n\right)\right)^{\Gamma} = \bigoplus_{ij} a^{\mathbf{4}}_{ij}\mathbb{C}^{N_i}\otimes\left(\mathbb{C}^{N_j}\right)^*,$$

   which means that these fermions in the fundamental $\mathbf{4}$ of the original R-symmetry now become $\left(N_i,\overline{N}_j\right)$ bi-fundamentals of G and there are $a^{\mathbf{4}}_{ij}$ copies of them;

3. $a^{\mathbf{6}}_{ij}$ scalars $\Phi^{ij}_{f_{ij}}$ $(f_{ij} = 1,\ldots,a^{\mathbf{6}}_{ij})$ as

$$\left(\mathbf{6}\otimes\hom\left(\mathbb{C}^n,\mathbb{C}^n\right)\right)^{\Gamma} = \bigoplus_{ij} a^{\mathbf{6}}_{ij}\mathbb{C}^{N_i}\otimes\left(\mathbb{C}^{N_j}\right)^*,$$

   similarly, these are G bi-fundamental bosons, inherited from the $\mathbf{6}$ of the original R-symmetry.

For the above, we define $a_{ij}^{\mathcal{R}}$ ($\mathcal{R} = \mathbf{4}$ or $\mathbf{6}$ for fermions and bosons respectively) as the composition coefficients

$$\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^{\mathcal{R}} \mathbf{r}_j \, . \tag{2.1}$$

Moreover, the supersymmetry of the projected theory must have its R-symmetry in the commutant of $\Gamma \subset SU(4)$, which is $U(2)$ for $SU(2)$, $U(1)$ for $SU(3)$ and trivial for $SU(4)$, which means: if $\Gamma \subset SU(2)$, we have an $\mathcal{N} = 2$ theory, if $\Gamma \subset SU(3)$, we have $\mathcal{N} = 1$, and finally for $\Gamma \subset$ the full $SU(4)$, we have a non-supersymmetric theory.

Taking the character χ for element $\gamma \in \Gamma$ on both sides of (2.1) and recalling that χ is a $(\otimes, \oplus)$-ring homomorphism, we have

$$\chi_\gamma^{\mathcal{R}} \chi_\gamma^{(i)} = \sum_{j=1}^{r} a_{ij}^{\mathcal{R}} \chi_\gamma^{(j)} \, , \tag{2.2}$$

where $r = |\{\mathbf{r}_i\}|$, the number of irreducible representations, which by an elementary theorem on finite characters, is equal to the number of inequivalent conjugacy classes of Γ. We further recall the orthogonality theorem of finite characters,

$$\sum_{\gamma=1}^{r} r_\gamma \chi_\gamma^{(i)*} \chi_\gamma^{(j)} = g \delta^{ij} \, , \tag{2.3}$$

where $g = |\Gamma|$ is the order of the group and r_γ is the order of the conjugacy class containing γ. Indeed, χ is a class function and is hence constant for each conjugacy class; moreover, $\sum\limits_{\gamma=1}^{r} r_\gamma = g$ is the class equation for Γ. This orthogonality allows us to invert (2.2) to finally give the matrix a_{ij} for the matter content

$$a_{ij}^{\mathcal{R}} = \frac{1}{g} \sum_{\gamma=1}^{r} r_\gamma \chi_\gamma^{\mathcal{R}} \chi_\gamma^{(i)} \chi_\gamma^{(j)*} \, , \tag{2.4}$$

where $\mathcal{R} = \mathbf{4}$ for Weyl fermions and $\mathbf{6}$ for adjoint scalars and the sum is effectively that over the columns of the Character Table of Γ. Thus equipped, let us specialise to Γ being finite discrete subgroups of $SU(2)$ and $SU((3)$.

3. Checks for $SU(2)$

The subgroups of $SU(2)$ have long been classified [19]; discussions and applications thereof can be found in [16, 17, 18, 22]. To algebraic geometers they give rise to the so-called Klein singularities and are labeled by the first affine extension of the simply-laced simple Lie groups $\widehat{A}\widehat{D}\widehat{E}$ (whose associated Dynkin diagrams are those of ADE adjointed by an extra node), i.e., there are two infinite series and 3 exceptional cases:

1. $\widehat{A}_n = \mathbb{Z}_{n+1}$, the cyclic group of order $n+1$;

2. $\widehat{D}_n$, the binary lift of the ordinary dihedral group d_n;

3. the three exceptional cases, $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$, the so-called binary or double[1] tetrahedral, octahedral and icosahedral groups $\mathcal{T}, \mathcal{O}, \mathcal{I}$.

The character tables for these groups are known [25, 26, 28] and are included in appendix A for reference. Therefore to obtain (2.4) the only difficulty remains in the choice of $\mathcal{R}$. We know that whatever $\mathcal{R}$ is, it must be 4 dimensional for the fermions and 6 dimensional for the bosons inherited from the fundamental **4** and antisymmetric **6** of $SU(4)$. Such an $\mathcal{R}$ must therefore be a 4 (or 6) dimensional irrep of Γ, or be the tensor sum of lower dimensional irreps (and hence be reducible); for the character table, this means that the row of characters for $\mathcal{R}$ (extending over the conjugacy classes of Γ) must be an existing row or the sum of existing rows. Now since the first column of the character table of any finite group precisely gives the dimension of the corresponding representation, it must therefore be that $\dim(\mathcal{R}) = 4, 6$ should be partitioned into these numbers. Out of these possibilities we must select the one(s) consistent with the decomposition of the **4** and **6** of $SU(4)$ into the $SU(2)$ subgroup[2], namely:

$$\begin{array}{rcl} SU(4) & \rightarrow & SU(2) \times SU(2) \times U(1)\,, \\ \mathbf{4} & \rightarrow & (\mathbf{2},\mathbf{1})_{+1} \bigoplus (\mathbf{1},\mathbf{2})_{-1}\,, \\ \mathbf{6} & \rightarrow & (\mathbf{1},\mathbf{1})_{+2} \bigoplus (\mathbf{1},\mathbf{1})_{-2} \bigoplus (\mathbf{2},\mathbf{2})_{0}\,, \end{array} \tag{3.1}$$

where the subscripts correspond to the $U(1)$ factors (i.e., the trace) and in particular the $\pm$ forces the overall traceless condition. From (3.1) we know that $\Gamma \subset SU(2)$ inherits a **2** while the complement is trivial. This means that the 4 dimensional represention of Γ must be decomposable into a nontrivial 2 dimensional one with a trivial 2 dimensional one. In the character language, this means that $\mathcal{R} = \mathbf{4} = \mathbf{2}_{\textbf{trivial}} \oplus \mathbf{2}$ where $\mathbf{2}_{\textbf{trivial}} = \mathbf{1}_{\textbf{trivial}} \oplus \mathbf{1}_{\textbf{trivial}}$, the tensor sum of two copies of the (trivial) principal representation where all group elements are mapped to the identity, i.e., corresponding to the first row in the character table. Whereas for the bosonic case

[1] For $SO(3) \cong SU(2)/\mathbb{Z}_2$ these would be the familiar symmetry groups of the respective regular solids in $\mathbb{R}^3$: the dihedron, tetrahedron, octahedron/cube and icosahedron/dodecahedron. However since we are in the double cover $SU(2)$, there is a non-trivial $\mathbb{Z}_2$- lifting,

$$\begin{array}{ccccccc} 0 \rightarrow \mathbb{Z}_2 \rightarrow & SU(2) & \rightarrow & SO(3) & \rightarrow 0\,, \\ & \bigcup & & \bigcup & \\ & \widehat{D}, \mathcal{T}, \mathcal{O}, \mathcal{I} & \rightarrow & d, T, O, I & \end{array}$$

hence the modifier "binary". Of course, the A-series, being abelian, receives no lifting.Later on we shall briefly touch upon the ordinary d, T, I, O groups as well.

[2] We note that even though this decomposition is that into irreducibles for the full continuous Lie groups, such irreducibility may not be inherited by the discrete subgroup, i.e., the **2**'s *may* not be irreducible representations of the finite Γ.

we have $\mathcal{R} = \mathbf{6} = \mathbf{2}_{\mathbf{trivial}} \oplus \mathbf{2} \oplus \mathbf{2}'$. We have denoted $\mathbf{2}'$ to signify that the two $\mathbf{2}$'s may not be the same, and correspond to inequivalent representations of Γ with the same dimension. However we can restrict this further by recalling that the antisymmetrised tensor product $[\mathbf{4} \otimes \mathbf{4}]_A \to 1 \oplus 2 \oplus 2 \oplus [2 \otimes 2]_A$ must in fact contain the $\mathbf{6}$. Whence we conclude that $\mathbf{2} = \mathbf{2}'$. Now let us again exploit the additive property of the group character, i.e., a homomorphism from a $\oplus$-ring to a $+$-subring of a number field (and indeed much work has been done for the subgroups in the case of number fields of various characteristics); this means that we can simplify $\chi^{\mathcal{R}=x\oplus y}$ as $\chi^x + \chi^y$. Consequently, our matter matrices become:

$$a^{\mathbf{4}}_{ij} = \frac{1}{g}\sum_{\gamma=1}^{r} r_\gamma \left(2\chi^{\mathbf{1}}_\gamma + \chi^{\mathbf{2}}_\gamma\right)\chi^{(i)}_\gamma \chi^{(j)*}_\gamma = 2\delta_{ij} + \frac{1}{g}\sum_{\gamma=1}^{r} r_\gamma \chi^{\mathbf{2}}_\gamma \chi^{(i)}_\gamma \chi^{(j)*}_\gamma ,$$
$$a^{\mathbf{6}}_{ij} = \frac{1}{g}\sum_{\gamma=1}^{r} r_\gamma \left(2\chi^{\mathbf{1}}_\gamma + \chi^{\mathbf{2}\oplus\mathbf{2}}_\gamma\right)\chi^{(i)}_\gamma \chi^{(j)*}_\gamma = 2\delta_{ij} + \frac{2}{g}\sum_{\gamma=1}^{r} r_\gamma \chi^{\mathbf{2}}_\gamma \chi^{(i)}_\gamma \chi^{(j)*}_\gamma ,$$

where we have used the fact that χ of the trivial representation are all equal to 1, thus giving by (2.3), the δ_{ij}'s. This simplification thus limits our attention to only 2 dimensional representations of Γ; however there still may remain many possibilities since the $\mathbf{2}$ may be decomposed into nontrivial $\mathbf{1}$'s or there may exist many inequivalent irreducible $\mathbf{2}$'s.

We now appeal to physics for further restriction. We know that the $\mathcal{N} = 2$ theory (which we recall is the resulting case when $\Gamma \subset SU(2)$) is a non-chiral supersymmetric theory; this means our bifundamental fields should not distinguish the left and right indices, i.e., the matter matrix a_{ij} must be *symmetric*. Also we know that in the $\mathcal{N} =$ 2 vector multiplet there are 2 Weyl fermions and 2 real scalars, thus the fermionic and bosonic matter matrices have the same entries on the diagonal. Furthermore the hypermultiplet has 2 scalars and 1 Weyl fermion in $(N_i, \bar{N}_j)$ and another 2 scalars and 1 Weyl fermion in the complex conjugate representation, whence we can restrict the off-diagonals as well, viz., $2a^{\mathbf{4}}_{ij} - a^{\mathbf{6}}_{ij}$ must be some multiple of the identity. This supersymmetry matching is of course consistent with (3).

Enough said on generalities. Let us analyse the groups case by case. For the cyclic group, the $\mathbf{2}$ must come from the tensor sum of two $\mathbf{1}$'s. Of all the possibilities, only the pairing of dual representations gives symmetric a_{ij}. By dual we mean the two $\mathbf{1}$'s which are complex conjugates of each other (this of course includes when $\mathbf{2} = \mathbf{1}^{\mathbf{2}}_{\mathbf{trivial}}$, which exist for all groups and gives us merely δ_{ij}'s and can henceforth be eliminated as uninteresting). We denote the nontrivial pairs as $\mathbf{1}'$ and $\mathbf{1}''$. In this case we can easily perform yet another consistency check. From (3.1), we have a traceless condition seen as the cancelation of the $U(1)$ factors. That was on the Lie algebra level; as groups, this is our familiar determinant unity condition. Since in the block decomposition (3.1) the $\mathbf{2}_{\mathbf{trivial}} \subset$ the complement $SU(4)\backslash\Gamma$ clearly has determinant 1, this forces our $\mathbf{2}$ matrix to have determinant 1 as well. However in this cyclic case, Γ is abelian, whence the characters are simply presentations of the group, making the $\mathbf{2}$ to be in fact diagonal. Thus the determinant is simply the product of the entries of the two rows in the character table. And indeed we see for dual representations, being complex conjugate roots of unity, the two rows do multiply to 1 for all members. Furthermore we note that different dual pairs give a_{ij}'s that are mere permutations of each other. We conclude that the fermion matrix arises from $\mathbf{1^2} \oplus \mathbf{1}^{'} \oplus \mathbf{1}^{''}$. For the bosonic matrix, by (3), we have $\mathbf{6} = (\mathbf{1} \oplus \mathbf{1}^{'} \oplus \mathbf{1}^{''})^{\mathbf{2}}$. These and ensuing a_{ij}'s are included in appendix B.

For the dihedral case, the $\mathbf{1}$'s are all dual to the principal, corresponding to some $\mathbb{Z}_2$ inner automorphism among the conjugacy classes and the characters consist no more than ± 1's, giving us a_{ij}'s which are block diagonal in $((1,0),(0,1))$ or $((0,1),(1,0))$ and are not terribly interesting. Let us rigorise this statement. Whenever we have the character table consisting of a row that is composed of cycles of roots of unity, which is a persistent theme for $\mathbf{1}$ irreps, this corresponds in general to some $\mathbb{Z}_k$ action on the conjugacy classes. This implies that our a_{ij} for this choice of $\mathbf{1}$ will be the Kronecker product of matrices obtained from the cyclic groups which offer us nothing new. We shall refer to these cases as "blocks"; they offer us another condition of elimination whose virtues we shall exploit much. In light of this, for the dihedral the choice of the $\mathbf{2}$ comes from the irreducible $\mathbf{2}$'s which again give symmetric a_{ij}'s that are permutations among themselves. Hence $\mathcal{R} = \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2}$ and $\mathcal{R} = \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2}$. For reference we have done likewise for the dihedral series not in the full $SU(2)$, the choice for $\mathcal{R}$ is the same for them.

Finally for the exceptionals $\mathcal{T}, \mathcal{O}, \mathcal{I}$, the $\mathbf{1}$'s again give uninteresting block diagonals and out choice of $\mathbf{2}$ is again unique up to permutation. Whence still $\mathcal{R} = \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2}$ and $\mathcal{R} = \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2}$. For reference we have computed the ordinary exceptionals T, O, I which live in $SU(2)$ with its center removed, i.e., in $SU(2)/\mathbb{Z}_2 \cong SO(3)$. For them the $\mathbf{2}$ comes from the $\mathbf{1}^{'} \oplus \mathbf{1}^{''}$, the $\mathbf{2}$, and the trivial $\mathbf{1^2}$ respectively.

Of course we can perform an *a posteriori* check. In this case of $SU(2)$ we already know the matter content due to the works on quiver diagrams [1, 17, 14]. The theory dictates that the matter content a_{ij} can be obtained by looking at the Dynkin diagram of the $\widehat{A}\widehat{D}\widehat{E}$ group associated to Γ whereby one assigns 2 for a_{ij} on the diagonal as well as 1 for every pair of connected nodes $i \to j$ and 0 otherwise, i.e., a_{ij} is essentially the adjacency matrix for the Dynkin diagrams treated as unoriented graphs. Of course adjacency matrices for unoriented graphs are symmetric; this is consistent with our nonchiral supersymmetry argument. Furthermore, the dimension of $a_{ij}^{\mathbf{4}}$ is required to be equal to the number of nodes in the associated affine Dynkin diagram (i.e., the rank). This property is immediately seen to be satisfied by examining the character tables in appendix A where we note that the number of conjugacy classes of the respective finite groups (which we recall is equal to the number of irreducible representations) and hence the dimension of a_{ij} is indeed that for the ranks of the associated affine algebras, namely $n+1$ for $\widehat{A_n}$ and $\widehat{D_n}$ and 7,8,9 for $\widehat{E_{6,7,8}}$ respectively.

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We note in passing that the conformality condition $N_f = 2N_c$ for this $\mathcal{N} = 2$ [4, 5] nicely translates to the graph language: it demands that for the one loop β-function to vanish the label of each node (the gauge fields) must be $\frac{1}{2}$ that of those connected thereto (the bi-fundamentals).

Our results for a_{ij} computed using (2.4), appendix A, and the aforementioned decomposition of $\mathcal{R}$ are tabulated in appendix B. They are precisely in accordance with the quiver theory and present themselves as the relevant adjacency matrices. One interesting point to note is that for the dihedral series, the ordinary d_n (which are in $SO(3)$ and not $SU(2)$) for even n also gave the binary $\widehat{D_{n'=\frac{n+6}{2}}}$ Dynkin diagram while the odd n case always gave the ordinary $D_{n'=\frac{n+3}{2}}$ diagram.

These results should be of no surprise to us, since a similar calculation was in fact done by J. Mckay when he first noted his famous correspondence [16]. In the paper he computed the composition coefficients m_{ij} in $R \otimes R_j = \bigoplus\limits_k m_{jk} R_k$ for $\Gamma \subset SU(2)$ with R being a faithful representation thereof. He further noted that for all these Γ's there exists (unique up to automorphism) such R, which is precisely the 2 dimensional irreducible representation for $\widehat{D}$ and $\widehat{E}$ whereas for $\widehat{A}$ it is the direct sum of a pair of dual 1 dimensional representations. Indeed this is exactly the decomposition of $\mathcal{R}$ which we have argued above from supersymmetry. His *Theorema Egregium* was then

Theorem 1. *The matrix m_{ij} is $2I$ minus the cartan matrix, and is thus the adjacency matrix for the associated affine Dynkin diagram treated as undirected C_2-graphs (i.e., maximal eigenvalue is 2).*

Whence m_{ij} has 0 on the diagonal and 1 for connected nodes. Now we note from our discussions above and results in appendix B, that our $\mathcal{R}$ is precisely Mckay's R (which we henceforth denote as R_M) plus two copies of the trivial representation for the **4** and R_M plus the two dimensional irreps in addition to the two copies of the trivial for the **6**. Therefore we conclude from (2.4):

$$a_{ij}^{\mathbf{4}} = \frac{1}{g} \sum_{\gamma=1}^{r} r_\gamma \chi_\gamma^{R_M \oplus \mathbf{1}^2} \chi_\gamma^{(i)} \chi_\gamma^{(j)*} , \qquad a_{ij}^{\mathbf{6}} = \frac{1}{g} \sum_{\gamma=1}^{r} r_\gamma \chi_\gamma^{R_M \oplus R_M \oplus \mathbf{1}^2} \chi_\gamma^{(i)} \chi_\gamma^{(j)*} ,$$

which implies of course, that our matter matrices should be

$$a_{ij}^{\mathbf{4}} = 2\delta_{ij} + m_{ij} , \qquad a_{ij}^{\mathbf{6}} = 2\delta_{ij} + 2m_{ij} ,$$

with Mckay's m_{ij} matrices. This is exactly the results we have in appendix B. Having obtained such an elegant graph-theoretic interpretation to our results, we remark that from this point of view, oriented graphs means chiral gauge theory and connected means interacting gauge theory. Hence we have the foresight that the $\mathcal{N} = 1$ case which we shall explore next will involve oriented graphs.

Now Mckay's theorem explains why the discrete subgroups of $SU(2)$ and hence Klein singularities of algebraic surfaces (which our orbifolds essentially are) as well as subsequent gauge theories thereupon afford this correspondence with the affine simply-laced Lie groups. However they were originally proven on a case by case basis, and we would like to know a deeper connection, especially in light of quiver theories. We can partially answer this question by noting a beautiful theorem due to Gabriel [20, 21] which forces the quiver considerations by Douglas et al. [1] to have the ADE results of Mckay.

It turns out to be convenient to formulate the theory axiomatically. We define $\mathcal{L}(\gamma, \Lambda)$, for a finite connected graph γ with orientation Λ, vertices γ_0 and edges γ_1, to be the category of quivers whose objects are any collection (V, f) of spaces $V_{\alpha\in\gamma_0}$ and mappings $f_{l\in\gamma_1}$ and whose morphisms are $\phi : (V, f) \to (V', f')$ a collection of linear mappings $\phi_{\alpha\in\Gamma_0} : V_\alpha \to V'_\alpha$ compatible with f by $\phi_{e(l)} f_l = f'_l \phi_{b(l)}$ where $b(l)$ and $e(l)$ are the beginning and end of the directed edge l. Then we have

Theorem 2. *If in the quiver category $\mathcal{L}(\gamma, \Lambda)$ there are only finitely many non-isomorphic indecomposable objects, then the graph γ coincides with one of the graphs $A_n, D_n, E_{6,7,8}$.* This theorem essentially compels any finite quiver theory to be constructible only on graphs which are of the type of the Dynkin diagrams of ADE. And indeed, the theories of Douglas, Moore et al. [1, 14] have explicitly made the physical realisations of these constructions. We therefore see how Mckay's calculations, quiver theory and our present calculations nicely fit together for the case of $\Gamma \subset SU(2)$.

4. The case for $SU(3)$

We repeat the above analysis for $\Gamma = SU(3)$, though now we have no quiver-type theories to aid us. The discrete subgroups of $SU(3)$ have also been long classified [22]. They include (the order of these groups are given by the subscript), other than all those of $SU(2)$ since $SU(2) \subset SU(3)$, the following new cases. We point out that in addition to the cyclic group in $SU(2)$, there is now in fact another Abelian case $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ for $SU(3)$ generated by the matrix $((e^{\frac{2\pi i}{k}}, 0, 0), (0, e^{\frac{2\pi i}{k'}}, 0), (0, 0, e^{-\frac{2\pi i}{k} - \frac{2\pi i}{k'}}))$ much in the spirit that $((e^{\frac{2\pi i}{n}}, 0), (0, e^{-\frac{2\pi i}{n}}))$ generates the $\mathbb{Z}_n$ for $SU(2)$. Much work has been done for this $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ case, q. v. [8] and references therein.

1. Two infinite series Δ_{3n^2} and Δ_{6n^2} for $n \in \mathbb{Z}$, which are analogues of the dihedral series in $SU(2)$:

    (a) $\Delta \subset$ only the full $SU(3)$: when $n = 0 \bmod 3$ where the number of classes for $\Delta(3n^2)$ is $(8 + \frac{1}{3}n^2)$ and for $\Delta(6n^2)$, $\frac{1}{6}(24 + 9n + n^2)$;

    (b) $\Delta \subset$ both the full $SU(3)$ and $SU(3)/\mathbb{Z}_3$: when $n \neq 0 \bmod 3$ where the number of classes for $\Delta(3n^2)$ is $\frac{1}{3}(8 + n^2)$ and for $\Delta(6n^2)$, $\frac{1}{6}(8 + 9n + n^2)$;

2. Analogues of the exceptional subgroups of $SU(2)$, and indeed like the later, there are two series depending on whether the $\mathbb{Z}_3$-center of $SU(3)$ has been modded out (we recall that the binary $\mathcal{T}, \mathcal{O}, \mathcal{I}$ are subgroups of $SU(2)$, while the ordinary T, O, I are subgroups of the center-removed $SU(2)$, i.e., $SO(3)$, and not the full $SU(2)$):

   (a) For $SU(3)/\mathbb{Z}_3$:
   $\Sigma_{36}, \Sigma_{60} \cong A_5$, the alternating symmetric-5 group, which incidentally is precisely the ordinary icosahedral group $I, \Sigma_{72}, \Sigma_{168} \subset S_7$, the symmetric-7 group, $\Sigma_{216} \supset \Sigma_{72} \supset \Sigma_{36}$, and $\Sigma_{360} \cong A_6$, the alternating symmetric-6 group;

   (b) For the full[3] $SU(3)$:
   $\Sigma_{36\times 3}, \Sigma_{60\times 3} \cong \Sigma_{60} \times \mathbb{Z}_3, \Sigma_{168\times 3} \cong \Sigma_{168} \times \mathbb{Z}_3, \Sigma_{216\times 3}$, and $\Sigma_{360\times 3}$.

Up-to-date presentations of these groups and some character tables may be found in [23, 24]. The rest have been computed with [27]. These are included in appendix C for reference. As before we must narrow down our choices for $\mathcal{R}$. First we note that it must be consistent with the decomposition:

$$\begin{array}{ccc} SU(4) & \to & SU(3) \times U(1) \\ \mathbf{4} & \to & \mathbf{3}_{-1} \bigoplus \mathbf{1}_3 \\ \mathbf{6} & \to & \mathbf{3}_2 \bigoplus \bar{\mathbf{3}}_{-2} \end{array} \qquad (4.1)$$

This decomposition (4.1), as in the comments for (3.1), forces us to consider only 3 dimensionals (possibly reducible) and for the fermion case the remaining $\mathbf{1}$ must in fact be the trivial, giving us a δ_{ij} in $a^{\mathbf{4}}_{ij}$.

Now as far as the symmetry of a_{ij} is concerned, since $SU(3)$ gives rise to an $\mathcal{N}=1$ chiral theory, the matter matrices are no longer necessarily symmetric and we can no longer rely upon this property to guide us. However we still have a matching condition between the bosons and the fermions. In this $\mathcal{N}=1$ chiral theory we have 2 scalars and a Weyl fermion in the chiral multiplet as well as a gauge field and a

[3] In his work on Gorenstein singularities [23], Yau points out that since the cases of $\Sigma_{60\times 3}$ and $\Sigma_{168\times 3}$ are simply direct products of the respective cases in $SU(3)/\mathbb{Z}_3$ with $\mathbb{Z}_3$, they are usually left out by most authors. The direct product simply extends the class equation of these groups by 3 copies and acts as an inner automorphism on each conjugacy class. Therefore the character table is that of the respective center-removed cases, but with the entries each multiplied by the matrix $((1,1,1),(1,w,w^2),(1,w^2,w))$ where $w = \exp(2\pi i/3)$, i.e., the full character table is the Kronecker product of that of the corresponding center-removed group with that of $\mathbb{Z}_3$. Subsequently, the matter matrices a_{ij} become the Kronecker product of a_{ij} for the center-removed groups with that for $\Gamma = \mathbb{Z}_3$ and gives no interesting new results. In light of this, we shall adhere to convention and call Σ_{60} and Σ_{168} subgroups of both $SU(3)/\mathbb{Z}_3$ and the full $SU(3)$ and ignore $\Sigma_{60\times 3}$ and $\Sigma_{168\times 3}$.

Weyl fermion in the vector multiplet. If we denote the chiral and vector matrices as C_{ij} and V_{ij}, and recalling that there is only one adjoint field in the vector multiplet, then we should have:

$$
\begin{aligned}
a^{\mathbf{4}}_{ij} &= V_{ij} + C_{ij} = \delta_{ij} + C_{ij}\ , \\
a^{\mathbf{6}}_{ij} &= C_{ij} + C_{ji}\ .
\end{aligned}
\tag{4.2}
$$

This decomposition is indeed consistent with (4.1); where the δ_{ij} comes from the principal $\mathbf{1}$ and the C_{ij} and C_{ji}, from dual pairs of $\mathbf{3}$; incidentally it also implies that the bosonic matrix should be symmetric and that dual $\mathbf{3}$'s should give matrices that are mutual transposes. Finally as we have discussed in the A_n case of $SU(2)$, if one is to compose only from 1 dimensional representations, then the rows of characters for these $\mathbf{1}$'s must multiply identically to 1 over all conjugacy classes. Our choices for $\mathcal{R}$ should thus be restricted by these general properties.

Once again, let us analyse the groups case by case. First the Σ series. For the members which belong to the center-removed $SU(2)$, as with the ordinary T, O, I of $SU(2)/\mathbb{Z}_2$, we expect nothing particularly interesting (since these do not have non-trivial 3 dimensional representations which in analogy to the non-trivial 2 dimensional irreps of $\widehat{D_n}$ and $\widehat{E_{6,7,8}}$ should be the ones to give interesting results). However, for completeness, we shall touch upon these groups, namely, $\Sigma_{36,72,216,360}$. Now the $\mathbf{3}$ in (4.1) must be composed of $\mathbf{1}$ and $\mathbf{2}$. The obvious choice is of course again the trivial one where we compose everything from only the principal $\mathbf{1}$ giving $4\delta_{ij}$ and $6\delta_{ij}$ for the fermionic and bosonic a_{ij} respectively. We at once note that this is the only possibility for Σ_{360}, since its first non-trivial representation is 5 dimensional. Hence this group is trivial for our purposes. For Σ_{36}, the $\mathbf{3}$ can come only from $\mathbf{1}$'s for which case our condition that the rows must multiply to 1 implies that $\mathbf{3} = \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_4$, or $\Gamma_1 \oplus \Gamma_2^2$, both of which give uninteresting blocks, in the sense of what we have discussed in section 2. For Σ_{72}, we similarly must have $\mathbf{3} = \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4$ or $\mathbf{1}\oplus$ the self-dual $\mathbf{2}$, both of which again give trivial blocks. Finally for Σ_{216}, whose conjugacy classes consist essentially of $\mathbb{Z}_3$-cycles in the 1 and 2 dimensional representations, the $\mathbf{3}$ comes from $\mathbf{1} \oplus \mathbf{2}$ and the dual $\mathbf{3}$, from $\mathbf{1} \oplus \mathbf{2}'$.

For the groups belonging to the full $SU(3)$, namely $\Sigma_{168,60,36\times3,216\times3,360\times3}$, the situation is clear: as to be expected in analogy to the $SU(2)$ case, there always exist dual pairs of $\mathbf{3}$ representations. The fermionic matrix is thus obtained by tensoring the trivial representations with one member from a pair selected in turn out of the various pairs, i.e., $\mathbf{1} \oplus \mathbf{3}$; and indeed we have explicitly checked that the others (i.e., $\mathbf{1} \oplus \mathbf{3}'$) are permutations thereof. On the other hand, the bosonic matrix is obtained from tensoring any choice of a dual pair $\mathbf{3} \oplus \mathbf{3}'$ and again we have explicitly checked that other dual pairs give rise to permutations. We may be tempted to construct the **3** out of the **1**'s and **2**'s which do exist for $\Sigma_{36\times 3,216\times 3}$, however we note that in these cases the **1** and **2** characters are all cycles of $\mathbb{Z}_3$'s which would again give uninteresting blocks. Thus we conclude still that for all these groups, $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$ while $\mathbf{6} = \mathbf{3}\oplus$ dual $\bar{\mathbf{3}}$. These choices are of course obviously in accordance with the decomposition (4.1) above. Furthermore, for the Σ groups that belong solely to the full $SU(3)$, the dual pair of **3**'s always gives matrices that are mutual transposes, consistent with the requirement in (4.2) that the bosonic matrix be symmetric.

Moving on to the two Δ series. We note[4], that for $n = 1$, $\Delta_3 \cong \mathbb{Z}_3$ and $\Delta_6 \cong d_6$ while for $n = 2$, $\Delta_{12} \cong T := E_6$ and $\Delta_{24} \cong O := E_7$. Again we note that for all $n > 1$ (we have already analysed the $n = 1$ case[5] for $\Gamma \subset SU(2)$), there exist the dual **3** and $\mathbf{3}'$ representations as in the $\Sigma \subset$ full $SU(3)$ above; this is expected of course since as noted before, all the Δ groups at least belong to the full $SU(3)$. Whence we again form the fermionic a_{ij} from $\mathbf{1} \oplus \mathbf{3}'$, giving a generically nonsymmetric matrix (and hence a good chiral theory), and the bosonic, from $\mathbf{3} \oplus \mathbf{3}'$, giving us always a symmetric matrix as required. We note in passing that when $n = 0$ mod 3, i.e,. when the group belongs to both the full and the center-removed $SU(3)$, the Δ_{3n^2} matrices consist of a trivial diagonal block and an L-shaped block. Moreover, all the Δ_{6n^2} matrices are block decomposable. We shall discuss the significances of this observation in the next section. Our analysis of the discrete subgroups of $SU(3)$ is now complete; the results are tabulated in appendix D.

5. Quiver theory? chiral gauge theories?

Let us digress briefly to make some mathematical observations. We recall that in the $SU(2)$ case the matter matrices a_{ij}, due to Mckay's theorem and Moore-Douglas quiver theories, are encoded as adjacency matrices of affine Dynkin diagrams considered as unoriented graphs as given by figures 1 and 2.

We are of course led to wonder whether, in analogy, the a_{ij} for $SU(3)$ present themselves as adjacency matrices for quiver diagrams associated to some *oriented* graph theory because the theory is chiral. This is very much in the spirit of recent works on extensions of Mckay correspondences by algebraic geometers [35, 36]. We here present these quiver graphs in figures 3, 4 and 5, hoping that it may be of academic interest.

[4]Though congruence in this case really means group isomorphisms, for our purposes since only the group characters concern us, in what follows we might use the term loosely to mean identical character tables.

[5]Of course for $\mathbb{Z}_3$, we must have a different choice for $\mathcal{R}$, in particular to get a good chiral model, we take the $\mathbf{3} = \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{1}'''$

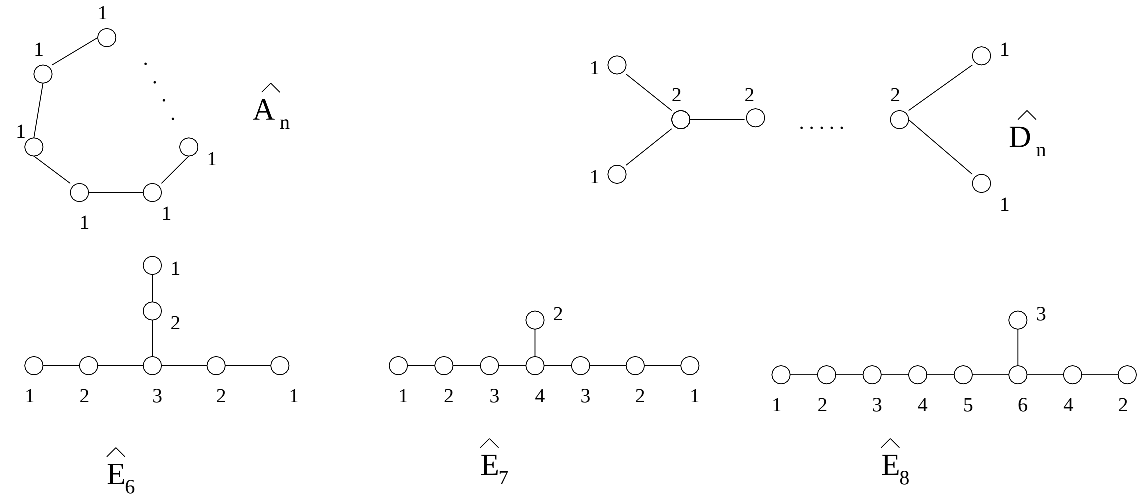

Figure 1: $\Gamma \subset$ full $SU(2)$ correspond to affine Dynkin diagrams with the Dynkin labels N_i on the nodes corresponding to the dimensions of the irreps. In the quiver theory the nodes correspond to gauge groups and the lines (or arrows for chiral theories), matter fields. For finite theories each N_i must be $\frac{1}{2}$ of the sum of neighbouring labels and the gauge group is $\bigoplus\limits_i U(N_i)$.

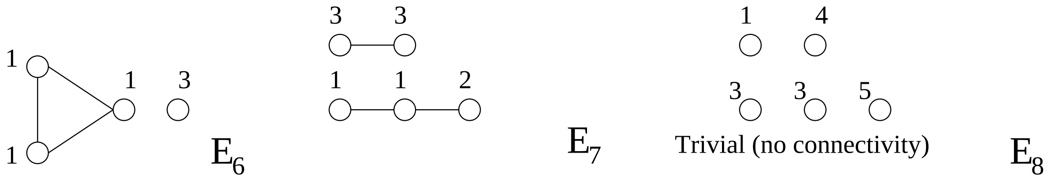

Figure 2: $\Gamma \subset SU(2)/\mathbb{Z}_2$ give disconnected graphs.

Indeed we note that for the center-removed case, as with $SU(2)$, we get disconnected (or trivial) graphs; this of course is the manifestation of the fact that there are no non-trivial **3** representations for these groups (just as there are no non-trivial **2**'s of $\Gamma \subset SU(2)/\mathbb{Z}_2$). On the other hand for $\Gamma \subset$ full $SU(3)$, we do get interesting connected and oriented graphs, composed of various directed triangular cycles.

Do we recognise these graphs? The answer is sort of yes and the right place to look for turns out to be in conformal field theory. In the work on general modular invariants in the WZW model for $\widehat{su(n)}_k$ (which is equivalent to the study of the modular properties of the characters for affine Lie algebras), an ADE classification was noted for $n = 2$ [28, 29, 30]; this should somewhat be expected due to our earlier discussion on Gabriel's Theorem. For $n = 3$, work has been done to extract coefficients in the fusion rules and to treat them as entries of adjacency matrices; this fundamentally is analogous to what we have done since fusion rules are an affine version of finite group composition coefficients. So-called generalised Dynkin diagrams have been constructed for $\widehat{su(3)}$ in analogy to the 5 simply-laced types

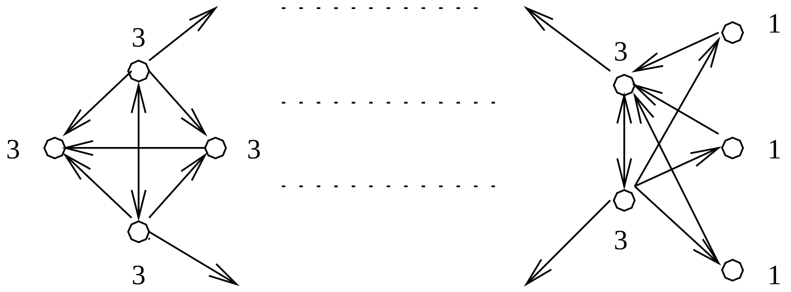

Figure 3: $\Delta_{3n^2} \subset SU(3)$ for $n \neq 0$ mod 3. These belong to both the full and center-removed $SU(3)$.

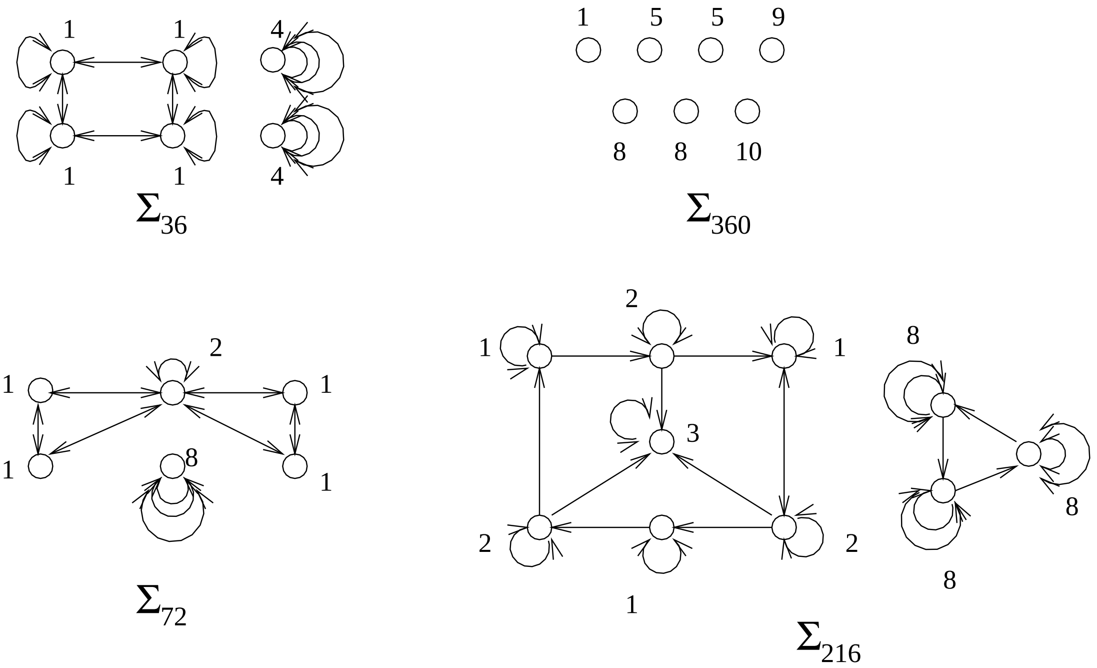

Figure 4: $\Sigma \subset SU(3)/\mathbb{Z}_3$ gives unconnected graphs.

corresponding to $SU(2)$, they are: $\mathcal{A}_n, \mathcal{D}_{3n}, \mathcal{E}_5, \mathcal{E}_9$ and $\mathcal{E}_{21}$ where the subscripts denote the level in the representation of the affine algebra [28, 31, 32]. We note a striking resemblance between these graphs (they are some form of a dual and we hope to rigorise this similarity in future work) with our quiver graphs: the $\mathcal{E}_5, \mathcal{E}_9$ and $\mathcal{E}_{21}$ correspond to $\Sigma_{216\times 3}$, $\Sigma_{360\times 3}$, and $\Sigma_{36\times 3}$ respectively. Incidentally these Σ groups are the *only* ones that belong solely to the full and not the center-removed $SU(3)$. The $\mathcal{D}_{3n}$ corresponds to Δ_{3n^2} for $n \neq 0$ mod 3, which are the non-trivial ones as observed in the previous section and which again are those that belong solely to the full $SU(3)$. The Δ_{6n^2} series, as noted above, gave non-connected graphs, and hence do not have a correspondent. Finally the $\mathcal{A}_n$, whose graph has complete $\mathbb{Z}_3$ symmetry must come from the Abelian subgroup of $SU(3)$, i.e., the $\widehat{A_n}$ case of $SU(2)$ but with $\mathcal{R} = \mathbf{3}$ and not $\mathbf{2}$.

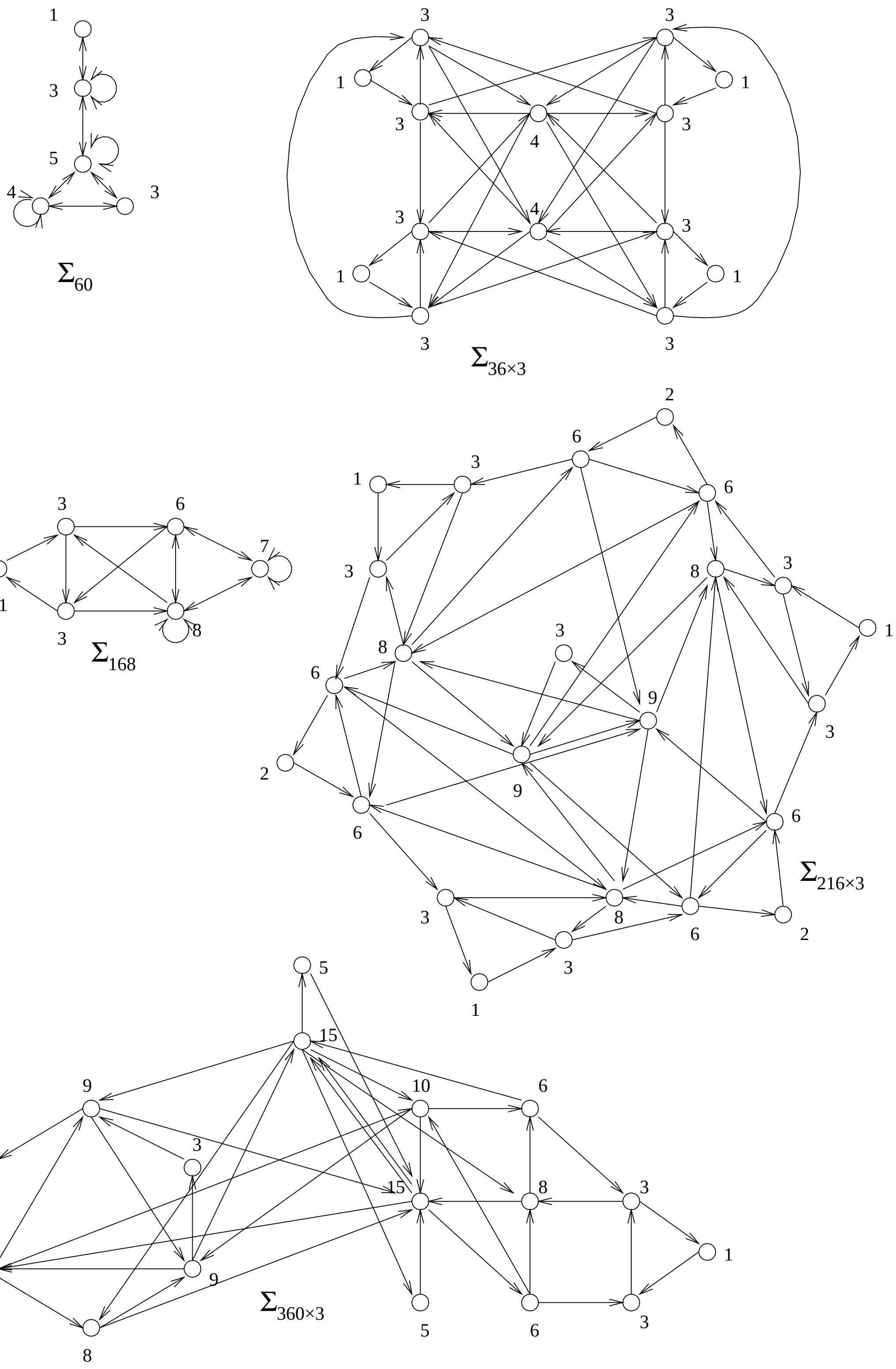

Figure 5: $\Sigma \subset$ full $SU(3)$. Only $\Sigma_{36\times3,216\times3,360\times3}$ belong only to the full $SU(3)$, for these we have the one loop β-function vanishing condition manifesting as the label of each node equaling to $\frac{1}{3}$ of that of the incoming and outgoing neighbours respectively. The matrix representation for these graphs are given in appendix D.

This beautiful relationship prompts us to make the following conjecture upon which we may labour in the near future:

Conjecture: *There exists a McKay-type correspondence between Gorenstein singularities and the characters of integrable representations of affine algebras* $\widehat{su(n)}$ *(and hence the modular invariants of the WZW model).*

A physical connection between $\widehat{SU(2)}$ modular invariants and quiver theories with 8 supercharges has been pointed out [34]. We remark that our conjecture is in the same spirit and a hint may come from string theory. If we consider a D1 string on our orbifold, then this is just our configuration of D3 branes after two T-dualities. In the strong coupling limit, this is just an F1 string in such a background which amounts to a non-linear sigma model and therefore some (super) conformal field theory whose partition function gives rise to the modular invariants. Moreover, connections between such modular forms and Fermat varieties have also been pointed out [33], this opens yet another door for us and many elegant intricacies arise.

Enough digression on mathematics; let us return to physics. We would like to conclude by giving a reference catalogue of chiral theories obtainable from $SU(3)$ orbifolds. Indeed, though some of the matrices may not be terribly interesting graph-theoretically, the non-symmetry of $a_{ij}^{\mathbf{4}}$ is still an indication of a good chiral theory.

For the original $U(n)$ theory it is conventional to take a canonical decomposition [5] as $n = N|\Gamma|$ [5], whence the (orbifolded) gauge group must be $\bigotimes\limits_i SU(N_i)$ as discussed in section 3, such that $N|\Gamma| = n = \sum\limits_i N_i|\mathbf{r}_i|$. By an elementary theorem on finite characters: $|\Gamma| = \sum\limits_i |\mathbf{r}_i|^2$, we see that the solution is $N_i = N|\mathbf{r}_i|$. This thus immediately gives the form of the gauge group. Incidentally for $SU(2)$, the McKay correspondence gives more information, it dictates that the dimensions of the irreps of Γ are actually the Dynkin labels for the diagrams. This is why we have labeled the nodes in the graphs above. Similarly for $SU(3)$, we have done so as well; these should be some form of generalised Dynkin labels.

Now for the promised catalogue, we shall list in table 1 all the chiral theories obtainable from orbifolds of $\Gamma \subset SU(3)$ ($\mathbb{Z}_3$ center-removed or not). This is done so by observing the graphs, connected or not, that contain unidirectional arrows. For completeness, we also include the subgroups of $SU(2)$, which are of course also in $SU(3)$, and which do give non-symmetric matter matrices (which we eliminated in the $\mathcal{N} = 2$ case) if we judiciously choose the $\mathbf{3}$ from their representations. We use the short hand $(n_1^{k_1}, n_2^{k_2}, \ldots, n_i^{k_i})$ to denote the gauge group $\bigoplus\limits^{k_1} SU(n_1) \cdots \bigoplus\limits^{k_i} SU(n_i)$. Analogous to the discussion in section 3, the conformality condition to one loop order in this $\mathcal{N} = 1$ case, viz., $N_f = 3N_c$ translates to the requirement that the label of each node must be $\frac{1}{3}$ of the sum of incoming and the sum of outgoing neighbours individually. (Incidentally, the gauge anomaly cancelation condition has been pointed out as well [9]. In our language it demands the restriction that $N_j a_{ij} = \bar{N}_j a_{ji}$.) In table 1, the * shall denote those groups for which this node condition is satisfied. We see that many of these models contain the group $SU(3) \times SU(2) \times U(1)$ and hope that some choice of orbifolds may thereby contain the Standard Model.

$\Gamma \subset SU(3)$	Gauge Group
$\widehat{A_n} \cong \mathbb{Z}_{n+1}$	(1^{n+1})
$\mathbb{Z}_k \times \mathbb{Z}_{k'}$	$(1^{kk'})*$
$\widehat{D_n}$	$(1^4, 2^{n-3})$
$\widehat{E_6} \cong \mathcal{T}$	$(1^3, 2^3, 3)$
$\widehat{E_7} \cong \mathcal{O}$	$(1^2, 2^2, 3^2, 4)$
$\widehat{E_8} \cong \mathcal{I}$	$(1, 2^2, 3^2, 4^2, 5, 6)$
$E_6 \cong T$	$(1^3, 3)$
$E_7 \cong O$	$(1^2, 2, 3^2)$
$E_8 \cong I$	$(1, 3^2, 4, 5)$
$\Delta_{3n^2}(n = 0 \bmod 3)$	$(1^9, 3^{\frac{n^2}{3}-1})*$
$\Delta_{3n^2}(n \neq 0 \bmod 3)$	$(1^3, 3^{\frac{n^2-1}{3}})*$
$\Delta_{6n^2}(n \neq 0 \bmod 3)$	$(1^2, 2, 3^{2(n-1)}, 6^{\frac{n^2-3n+2}{6}})*$
Σ_{168}	$(1, 3^2, 6, 7, 8)*$
Σ_{216}	$(1^3, 2^3, 3, 8^3)$
$\Sigma_{36\times 3}$	$(1^4, 3^8, 4^2)*$
$\Sigma_{216\times 3}$	$(1^3, 2^3, 3^7, 6^6, 8^3, 9^2)*$
$\Sigma_{360\times 3}$	$(1, 3^4, 5^2, 6^2, 8^2, 9^3, 10, 15^2)*$

Table 1: Chiral theories obtainable from orbifolds of $\Gamma \subset SU(3)$.

6. Concluding remarks

By studying gauge theories constructed from orbifolding of an $\mathcal{N} = 4$ $U(n)$ super-Yang-Mills theory in 4 dimensions, we have touched upon many issues. We have presented the explicit matter content and gauge group that result from such a procedure, for the cases of $SU(2)$ and $SU(3)$. In the first we have shown how our calculations agree with current quiver constructions and in the second we have constructed possible candidates for chiral theories. Furthermore we have noted beautiful graph-theoretic interpretations of these results: in the $SU(2)$ we have used Gabriel's theorem to partially explain the ADE outcome and in the $SU(3)$ we have noted connections with generalised Dynkin diagrams and have conjectured the existence of a McKay-type correspondence between these orbifold theories and modular invariants of WZW conformal models.

Much work of course remains. In addition to proving this conjecture, we also have numerous questions in physics. What about $SU(4)$, the full group? These would give interesting non-supersymmetric theories. How do we construct the brane box version of these theories? Roan has shown how the Euler character of these orbifolds correspond to the class numbers [36]; we know the blow-up of these singularities correspond to marginal operators. Can we extract the marginal couplings and thus the duality group this way? We shall hope to address these problems in forthcoming work. Perhaps after all, string orbifolds, gauge theories, modular invariants of conformal field theories as well as Gorenstein singularities and representations of affine Lie algebras, are all manifestations of a fundamental truism.

Acknowledgments

Ad Catharinæ Sanctæ Alexandriæ et ad majorem Dei gloriam...
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A. Character tables for the discrete subgroups of $SU(2)$

Henceforth we shall use Γ_i to index the representations and the numbers in the first row of the character tables shall refer to the order of each conjugacy class, or what we called r_γ.

$\widehat{A}_n$ = Cyclic $\mathbb{Z}_{n+1}$

	1	1	1	$\cdots$	1
Γ_1	1	1	1	$\cdots$	1
Γ_2	1	ϵ	ϵ^2	$\cdots$	ϵ^n
Γ_3	1	ϵ^2	ϵ^4	$\cdots$	ϵ^{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
Γ_n	1	ϵ^n	ϵ^{2n}	$\cdots$	ϵ^{n^2}

$\epsilon = \exp\left(\frac{2\pi i}{n+1}\right)$

For reference, next to each of the binary groups, we shall also include the character table of the corresponding ordinary cases, which are in $SU(2)/\mathbb{Z}_2$.

$\widehat{D}_n$ = Binary Dihedral

	1	1	2	$\cdots$	2	n	n
Γ_1	1	1	1	$\cdots$	1	1	1
Γ_2	1	1	1	$\cdots$	1	-1	-1
Γ_3	1	$(-1)^n$	$(-1)^1$	$\cdots$	$(-1)^{n-1}$	i^n	$-i^n$
Γ_4	1	$(-1)^n$	$(-1)^1$	$\cdots$	$(-1)^{n-1}$	$-i^n$	i^n
Γ_5	2	$(-2)^1$	$2\cos\frac{\pi}{n}$	$\cdots$	$2\cos\frac{\pi(n-1)}{n}$	0	0
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Γ_{n+1}	2	$(-2)^{n-1}$	$2\cos\frac{\pi(n-1)}{n}$	$\cdots$	$2\cos\frac{\pi(n-1)^2}{n}$	0	0

$\widehat{D}_{n'}$ = Dihedral D_n ($n' = \frac{n+3}{2}$ for odd n and $n' = \frac{n+6}{2}$ for even n)

	1	2	2	$\cdots$	2	n
Γ_1	1	1	1	$\cdots$	1	1
Γ_2	1	1	1	$\cdots$	1	-1
Γ_3	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos m\phi$	0
Γ_4	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2m\phi$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{n+3}{2}}$	2	$2\cos m\phi$	$2\cos 2m\phi$	$\cdots$	$2\cos m^2\phi$	0

n odd
$m = \frac{n-1}{2}$
$\phi = \frac{2\pi}{n}$

	1	2	2	$\cdots$	2	1	$n/2$	$n/2$
Γ_1	1	1	1	$\cdots$	1	1	1	1
Γ_2	1	1	1	$\cdots$	1	1	-1	-1
Γ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
Γ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
Γ_5	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos(m-1)\phi$	$2\cos m\phi$	0	0
Γ_6	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2(m-1)\phi$	$2\cos 2m\phi$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{n+6}{2}}$	2	$2\cos(m-1)\phi$	$2\cos 2(m-1)\phi$	$\cdots$	$2\cos(m-1)^2\phi$	$2\cos m(m-1)\phi$	0	0

n even
$m = \frac{n}{2}$
$\phi = \frac{2\pi}{n}$

$\widehat{E}_6$ = Binary Tetrahedral $\mathcal{T}$

	1	1	6	4	4	4	4
Γ_1	1	1	1	1	1	1	1
Γ_2	1	1	1	w	w	w^2	w^2
Γ_3	1	1	1	w^2	w^2	w	w
Γ_4	2	-2	0	1	-1	1	-1
Γ_5	2	-2	0	w	$-w$	w^2	$-w^2$
Γ_6	2	-2	0	w^2	$-w^2$	w	$-w$
Γ_7	3	3	-1	0	0	0	0

Ordinary T

	1	3	4	4
Γ_1	1	1	1	1
Γ_2	1	1	$-w$	w^2
Γ_3	1	1	w^2	$-w$
Γ_4	3	-1	0	0

$w = exp(\frac{2\pi i}{3})$

$\widehat{E}_7$ = Binary Octahedral $\mathcal{O}$

	1	1	2	6	6	8	8	16
Γ_1	1	1	1	1	1	1	1	1
Γ_2	1	1	-3	-1	-1	1	1	0
Γ_3	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	1	-1	0
Γ_4	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	1	-1	0
Γ_5	2	2	-2	0	0	-1	-1	1
Γ_6	3	3	-1	1	1	0	0	-1
Γ_7	3	3	-3	-1	-1	0	0	0
Γ_8	4	-4	0	0	0	-1	1	0

Ordinary O

	1	3	6	6	8
Γ_1	1	1	1	1	1
Γ_2	1	1	-1	-1	1
Γ_3	2	2	0	0	-1
Γ_4	3	-1	-1	1	0
Γ_5	3	-1	1	-1	0

$\widehat{E}_8$ = Binary Icosahedral $\mathcal{I}$

	1	1	12	12	12	12	20	20	30
Γ_1	1	1	1	1	1	1	1	1	1
Γ_2	2	-2	$-\bar{a}$	-1	a	$\bar{a}$	1	-1	0
Γ_3	2	-2	$-a$	$-\bar{a}$	$\bar{a}$	a	1	-1	0
Γ_4	3	3	$\bar{a}$	a	a	$\bar{a}$	0	0	-1
Γ_5	3	3	a	$\bar{a}$	$\bar{a}$	a	0	0	-1
Γ_6	4	-4	-1	-1	1	1	-1	1	0
Γ_7	4	4	-1	-1	-1	1	1	1	1
Γ_8	5	5	0	0	0	0	-1	-1	1
Γ_9	6	-6	1	1	-1	-1	0	0	0

Ordinary I

	1	12	12	15	20
Γ_1	1	1	1	1	1
Γ_2	3	a	$\bar{a}$	-1	0
Γ_3	3	$\bar{a}$	a	-1	0
Γ_4	4	-1	-1	0	1
Γ_5	5	0	0	1	-1

$a = \frac{1+\sqrt{5}}{2}$
$\bar{a} = \frac{1-\sqrt{5}}{2}$

B. Matter content for $\mathcal{N} = 2$ SUSY gauge theory ($\Gamma \subset SU(2)$)

Only the fermionic matrices are presented here; as can be seen from the decomposition, twice the fermion a_{ij} subtracted by $2\delta_{ij}$ should give the bosonic counterparts, which follows from supersymmetry. In the ensuing, **1** shall denote the (trivial) principal representation, $\mathbf{1}'$ and $\mathbf{1}''$, dual (conjugate) pairs of 1 dimensional representations.

$$\widehat{A}_n \quad \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1^2} \bigoplus \mathbf{1'} \bigoplus \mathbf{1''} \\ \mathbf{6} = \mathbf{1^2} \bigoplus \mathbf{1'^2} \bigoplus \mathbf{1''^2} \end{array}$$

$$\widehat{D}_{n'=\frac{n+6}{2}} \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 \end{pmatrix} \quad D_{n'=\frac{n+3}{2}} \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \bigoplus \mathbf{2} \\ \mathbf{6} = \mathbf{1}^2 \bigoplus \mathbf{2}^2 \end{array}$$

$$\widehat{E}_6 \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \bigoplus \mathbf{2} \\ \mathbf{6} = \mathbf{1}^2 \bigoplus \mathbf{2}^2 \end{array} \qquad \widehat{E}_7 \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \bigoplus \mathbf{2} \\ \mathbf{6} = \mathbf{1}^2 \bigoplus \mathbf{2}^2 \end{array}$$

$$\widehat{E}_8 \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \bigoplus \mathbf{2} \\ \mathbf{6} = \mathbf{1}^2 \bigoplus \mathbf{2}^2 \end{array}$$

C. Classification of discrete subgroups of $SU(3)$

C.1 Type I: the Σ series

These are the analogues of the $SU(2)$ crystallographic groups and their double covers, i.e., the E series. We have:

$$\Sigma_{36}, \Sigma_{72}, \Sigma_{216}, \Sigma_{60}, \Sigma_{168}, \Sigma_{360} \subset SU(3)/(\mathbb{Z}_3 \text{ center})$$
$$\Sigma_{36\times 3}, \Sigma_{72\times 3}, \Sigma_{216\times 3}, \Sigma_{60\times 3}, \Sigma_{168\times 3}, \Sigma_{360\times 3} \subset SU(3)$$

Type Ia: $\Sigma \subset SU(3)/\mathbb{Z}_3$

The character tables for the center-removed case have been given by [24].

Σ_{36}

	1	9	9	9	4	4
Γ_1	1	1	1	1	1	1
Γ_2	1	-1	-1	1	1	1
Γ_3	1	i	$-i$	-1	1	1
Γ_4	1	$-i$	i	-1	1	1
Γ_5	4	0	0	0	-2	1
Γ_6	4	0	0	0	1	-2

Σ_{72}

	1	18	18	18	9	8
Γ_1	1	1	1	1	1	1
Γ_2	1	1	-1	-1	1	1
Γ_3	1	-1	1	-1	1	1
Γ_4	1	-1	-1	1	1	1
Γ_5	2	0	0	0	-2	2
Γ_6	8	0	0	0	0	-1

Hessian Group
$\Sigma_{216} \supset \Sigma_{72} \supset \Sigma_{36}$

	1	12	12	54	36	36	9	8	24	24
Γ_1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	w	w^2	1	w	w^2	1	1	w	w^2
Γ_3	1	w^2	w	1	w^2	w	1	1	w^2	w
Γ_4	2	-1	-1	0	1	1	-2	2	-1	-1
Γ_5	2	$-w$	$-w^2$	0	w	w^2	-2	2	$-w$	$-w^2$
Γ_6	2	$-w^2$	$-w$	0	w^2	w	-2	2	$-w^2$	$-w$
Γ_7	3	0	0	-1	0	0	3	3	0	0
Γ_8	8	2	2	0	0	0	0	-1	-1	-1
Γ_9	8	$2w$	$2w^2$	0	0	0	0	-1	$-w$	$-w^2$
Γ_{10}	8	$2w^2$	$2w$	0	0	0	0	-1	$-w^2$	$-w$

$w = \exp \frac{2\pi i}{3}$

$\Sigma_{360} \cong A_6$

	1	40	45	72	72	90	40
Γ_1	1	1	1	1	1	1	1
Γ_2	5	2	1	0	0	-1	-1
Γ_3	5	-1	1	0	0	-1	2
Γ_4	8	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1
Γ_5	8	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1
Γ_6	9	0	1	-1	-1	1	0
Γ_7	10	1	-2	0	0	0	1

Type Ib: $\Sigma \subset$ **full** $SU(3)$

The character tables are computed, using [27], from the generators presented in [23]. In what follows, we define $e_n = \exp \frac{2\pi i}{n}$.

$\Sigma_{36\times 3}$

	1	12	1	12	9	9	1	9	9	9	9	9	9	9
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1
Γ_3	1	1	1	1	e_4	e_4	1	e_4	-1	-1	-1	$-e_4$	$-e_4$	$-e_4$
Γ_4	1	1	1	1	$-e_4$	$-e_4$	1	$-e_4$	-1	-1	-1	e_4	e_4	e_4
Γ_5	3	0	$3e_3^2$	0	1	e_3	$3e_3$	e_3^2	-1	$-e_3^2$	$-e_3$	1	e_3^2	e_3
Γ_6	3	0	$3e_3$	0	1	e_3^2	$3e_3^2$	e_3	-1	$-e_3$	$-e_3^2$	1	e_3	e_3^2
Γ_7	3	0	$3e_3^2$	0	-1	$-e_3$	$3e_3$	$-e_3^2$	-1	$-e_3^2$	$-e_3$	-1	$-e_3^2$	$-e_3$
Γ_8	3	0	$3e_3$	0	-1	$-e_3^2$	$3e_3^2$	$-e_3$	-1	$-e_3$	$-e_3^2$	-1	$-e_3$	$-e_3^2$
Γ_9	3	0	$3e_3^2$	0	e_4	e_{12}^7	$3e_3$	e_{12}^{11}	1	e_3^2	e_3	$-e_4$	$-e_{12}^{11}$	$-e_{12}^7$
Γ_{10}	3	0	$3e_3$	0	e_4	e_{12}^{11}	$3e_3^2$	e_{12}^7	1	e_3	e_3^2	$-e_4$	$-e_{12}^7$	$-e_{12}^{11}$
Γ_{11}	3	0	$3e_3^2$	0	$-e_4$	$-e_{12}^7$	$3e_3$	$-e_{12}^{11}$	1	e_3^2	e_3	e_4	e_{12}^{11}	e_{12}^7
Γ_{12}	3	0	$3e_3$	0	$-e_4$	$-e_{12}^{11}$	$3e_3^2$	$-e_{12}^7$	1	e_3	e_3^2	e_4	e_{12}^7	e_{12}^{11}
Γ_{13}	4	1	4	-2	0	0	4	0	0	0	0	0	0	0
Γ_{14}	4	-2	4	1	0	0	4	0	0	0	0	0	0	0

$\Sigma_{216\times 3}$

	1	24	1	12	12	54	54	1	12	54	72	12
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	${e_3}^2$	${e_3}^2$	1	1	1	${e_3}^2$	1	${e_3}^2$	e_3
Γ_3	1	1	1	e_3	e_3	1	1	1	e_3	1	e_3	${e_3}^2$
Γ_4	2	2	2	-1	-1	0	0	2	-1	0	-1	-1
Γ_5	2	2	2	$-e_3$	$-e_3$	0	0	2	$-e_3$	0	$-e_3$	$-{e_3}^2$
Γ_6	2	2	2	$-{e_3}^2$	$-{e_3}^2$	0	0	2	$-{e_3}^2$	0	$-{e_3}^2$	$-e_3$
Γ_7	3	3	3	0	0	-1	-1	3	0	-1	0	0
Γ_8	3	0	$3{e_3}^2$	$-{e_9}^2 + {e_9}^5$	$-{e_9}^2 - 2{e_9}^5$	1	e_3	$3e_3$	$2{e_9}^2 + {e_9}^5$	${e_3}^2$	0	$-2{e_9}^4 - {e_9}^7$
Γ_9	3	0	$3{e_3}^2$	$-{e_9}^2 - 2{e_9}^5$	$2{e_9}^2 + {e_9}^5$	1	e_3	$3e_3$	$-{e_9}^2 + {e_9}^5$	${e_3}^2$	0	${e_9}^4 + 2{e_9}^7$
Γ_{10}	3	0	$3{e_3}^2$	$2{e_9}^2 + {e_9}^5$	$-{e_9}^2 + {e_9}^5$	1	e_3	$3e_3$	$-{e_9}^2 - 2{e_9}^5$	${e_3}^2$	0	${e_9}^4 - {e_9}^7$
Γ_{11}	3	0	$3e_3$	${e_9}^4 - {e_9}^7$	$-2{e_9}^4 - {e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	${e_9}^4 + 2{e_9}^7$	e_3	0	$-{e_9}^2 - 2{e_9}^5$
Γ_{12}	3	0	$3e_3$	$-2{e_9}^4 - {e_9}^7$	${e_9}^4 + 2{e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	${e_9}^4 - {e_9}^7$	e_3	0	$2{e_9}^2 + {e_9}^5$
Γ_{13}	3	0	$3e_3$	${e_9}^4 + 2{e_9}^7$	${e_9}^4 - {e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	$-2{e_9}^4 - {e_9}^7$	e_3	0	$-{e_9}^2 + {e_9}^5$
Γ_{14}	6	0	$6{e_3}^2$	${e_9}^2 - {e_9}^5$	${e_9}^2 + 2{e_9}^5$	0	0	$6e_3$	$-2{e_9}^2 - {e_9}^5$	0	0	$2{e_9}^4 + {e_9}^7$
Γ_{15}	6	0	$6{e_3}^2$	${e_9}^2 + 2{e_9}^5$	$-2{e_9}^2 - {e_9}^5$	0	0	$6e_3$	${e_9}^2 - {e_9}^5$	0	0	$-{e_9}^4 - 2{e_9}^7$
Γ_{16}	6	0	$6{e_3}^2$	$-2{e_9}^2 - {e_9}^5$	${e_9}^2 - {e_9}^5$	0	0	$6e_3$	${e_9}^2 + 2{e_9}^5$	0	0	$-{e_9}^4 + {e_9}^7$
Γ_{17}	6	0	$6e_3$	$-{e_9}^4 + {e_9}^7$	$2{e_9}^4 + {e_9}^7$	0	0	$6{e_3}^2$	$-{e_9}^4 - 2{e_9}^7$	0	0	${e_9}^2 + 2{e_9}^5$
Γ_{18}	6	0	$6e_3$	$2{e_9}^4 + {e_9}^7$	$-{e_9}^4 - 2{e_9}^7$	0	0	$6{e_3}^2$	$-{e_9}^4 + {e_9}^7$	0	0	$-2{e_9}^2 - {e_9}^5$
Γ_{19}	6	0	$6e_3$	$-{e_9}^4 - 2{e_9}^7$	$-{e_9}^4 + {e_9}^7$	0	0	$6{e_3}^2$	$2{e_9}^4 + {e_9}^7$	0	0	${e_9}^2 - {e_9}^5$
Γ_{20}	8	-1	8	2	2	0	0	8	2	0	-1	2
Γ_{21}	8	-1	8	$2e_3$	$2e_3$	0	0	8	$2e_3$	0	$-e_3$	$2{e_3}^2$
Γ_{22}	8	-1	8	$2{e_3}^2$	$2{e_3}^2$	0	0	8	$2{e_3}^2$	0	$-{e_3}^2$	$2e_3$
Γ_{23}	9	0	$9{e_3}^2$	0	0	-1	$-e_3$	$9e_3$	0	$-{e_3}^2$	0	0
Γ_{24}	9	0	$9e_3$	0	0	-1	$-{e_3}^2$	$9{e_3}^2$	0	$-e_3$	0	0

$\cdots$

…

	12	9	9	9	12	72	36	36	36	36	36	36
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	e_3	1	1	1	e_3	e_3	e_3	e_3	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3
Γ_3	${e_3}^2$	1	1	1	${e_3}^2$	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3	e_3	e_3	${e_3}^2$
Γ_4	-1	-2	-2	-2	-1	-1	1	1	1	1	1	1
Γ_5	$-{e_3}^2$	-2	-2	-2	$-{e_3}^2$	$-{e_3}^2$	${e_3}^2$	${e_3}^2$	e_3	e_3	e_3	${e_3}^2$
Γ_6	$-e_3$	-2	-2	-2	$-e_3$	$-e_3$	e_3	e_3	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3
Γ_7	0	3	3	3	0	0	0	0	0	0	0	0
Γ_8	${e_9}^4 + 2{e_9}^7$	-1	$-{e_3}^2$	$-e_3$	${e_9}^4 - {e_9}^7$	0	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^7$
Γ_9	${e_9}^4 - {e_9}^7$	-1	$-{e_3}^2$	$-e_3$	$-2{e_9}^4 - {e_9}^7$	0	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^4$
Γ_{10}	$-2{e_9}^4 - {e_9}^7$	-1	$-{e_3}^2$	$-e_3$	${e_9}^4 + 2{e_9}^7$	0	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^2$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	${e_9}^4 + {e_9}^7$
Γ_{11}	$2{e_9}^2 + {e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$-{e_9}^2 + {e_9}^5$	0	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^2$
Γ_{12}	$-{e_9}^2 + {e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$-{e_9}^2 - 2{e_9}^5$	0	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^5$
Γ_{13}	$-{e_9}^2 - 2{e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$2{e_9}^2 + {e_9}^5$	0	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^7$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	${e_9}^2 + {e_9}^5$
Γ_{14}	$-{e_9}^4 - 2{e_9}^7$	2	$2{e_3}^2$	$2e_3$	$-{e_9}^4 + {e_9}^7$	0	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^7$
Γ_{15}	$-{e_9}^4 + {e_9}^7$	2	$2{e_3}^2$	$2e_3$	$2{e_9}^4 + {e_9}^7$	0	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^4$
Γ_{16}	$2{e_9}^4 + {e_9}^7$	2	$2{e_3}^2$	$2e_3$	$-{e_9}^4 - 2{e_9}^7$	0	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^2$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	${e_9}^4 + {e_9}^7$
Γ_{17}	$-2{e_9}^2 - {e_9}^5$	2	$2e_3$	$2{e_3}^2$	${e_9}^2 - {e_9}^5$	0	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^2$
Γ_{18}	${e_9}^2 - {e_9}^5$	2	$2e_3$	$2{e_3}^2$	${e_9}^2 + 2{e_9}^5$	0	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^5$
Γ_{19}	${e_9}^2 + 2{e_9}^5$	2	$2e_3$	$2{e_3}^2$	$-2{e_9}^2 - {e_9}^5$	0	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^7$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	${e_9}^2 + {e_9}^5$
Γ_{20}	2	0	0	0	2	-1	0	0	0	0	0	0
Γ_{21}	$2{e_3}^2$	0	0	0	$2{e_3}^2$	$-{e_3}^2$	0	0	0	0	0	0
Γ_{22}	$2e_3$	0	0	0	$2e_3$	$-e_3$	0	0	0	0	0	0
Γ_{23}	0	-3	$-3{e_3}^2$	$-3e_3$	0	0	0	0	0	0	0	0
Γ_{24}	0	-3	$-3e_3$	$-3{e_3}^2$	0	0	0	0	0	0	0	0

$\Sigma_{360\times 3}$

	1	72	72	72	72	72	72
Γ_1	1	1	1	1	1	1	1
Γ_2	3	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^2 - {e_{15}}^8$
Γ_3	3	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^2 - {e_{15}}^8$	$-{e_{15}}^{11} - {e_{15}}^{14}$
Γ_4	3	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^2 - {e_{15}}^8$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$
Γ_5	3	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_{15}}^2 - {e_{15}}^8$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-e_{15} - {e_{15}}^4$
Γ_6	5	0	0	0	0	0	0
Γ_7	5	0	0	0	0	0	0
Γ_8	6	1	1	${e_3}^2$	${e_3}^2$	e_3	e_3
Γ_9	6	1	1	e_3	e_3	${e_3}^2$	${e_3}^2$
Γ_{10}	8	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$
Γ_{11}	8	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$
Γ_{12}	9	-1	-1	-1	-1	-1	-1
Γ_{13}	9	-1	-1	$-{e_3}^2$	$-{e_3}^2$	$-e_3$	$-e_3$
Γ_{14}	9	-1	-1	$-e_3$	$-e_3$	$-{e_3}^2$	$-{e_3}^2$
Γ_{15}	10	0	0	0	0	0	0
Γ_{16}	15	0	0	0	0	0	0
Γ_{17}	15	0	0	0	0	0	0

…

…

	1	1	120	120	45	45	90	90	45	90
Γ_1	1	1	1	1	1	1	1	1	1	1
Γ_2	$3{e_3}^2$	$3e_3$	0	0	$-e_3$	$-{e_3}^2$	${e_3}^2$	e_3	-1	1
Γ_3	$3{e_3}^2$	$3e_3$	0	0	$-e_3$	$-{e_3}^2$	${e_3}^2$	e_3	-1	1
Γ_4	$3e_3$	$3{e_3}^2$	0	0	$-{e_3}^2$	$-e_3$	e_3	${e_3}^2$	-1	1
Γ_5	$3e_3$	$3{e_3}^2$	0	0	$-{e_3}^2$	$-e_3$	e_3	${e_3}^2$	-1	1
Γ_6	5	5	2	-1	1	1	-1	-1	1	-1
Γ_7	5	5	-1	2	1	1	-1	-1	1	-1
Γ_8	$6{e_3}^2$	$6e_3$	0	0	$2e_3$	$2{e_3}^2$	0	0	2	0
Γ_9	$6e_3$	$6{e_3}^2$	0	0	$2{e_3}^2$	$2e_3$	0	0	2	0
Γ_{10}	8	8	-1	-1	0	0	0	0	0	0
Γ_{11}	8	8	-1	-1	0	0	0	0	0	0
Γ_{12}	9	9	0	0	1	1	1	1	1	1
Γ_{13}	$9{e_3}^2$	$9e_3$	0	0	e_3	${e_3}^2$	${e_3}^2$	e_3	1	1
Γ_{14}	$9e_3$	$9{e_3}^2$	0	0	${e_3}^2$	e_3	e_3	${e_3}^2$	1	1
Γ_{15}	10	10	1	1	-2	-2	0	0	-2	0
Γ_{16}	$15{e_3}^2$	$15e_3$	0	0	$-e_3$	$-{e_3}^2$	$-{e_3}^2$	$-e_3$	-1	-1
Γ_{17}	$15e_3$	$15{e_3}^2$	0	0	$-{e_3}^2$	$-e_3$	$-e_3$	$-{e_3}^2$	-1	-1

Type Ic: $\Sigma \subset$ both $SU(3)$ and $SU(3)/\mathbb{Z}_3$

$\Sigma_{60} \cong A_5 \cong I$

	1	20	15	12	12
Γ_1	1	1	1	1	1
Γ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Γ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
Γ_4	4	1	0	-1	-1
Γ_5	5	-1	1	0	0

$\Sigma_{168} \subset S_7$

	1	21	42	56	24	24
Γ_1	1	1	1	1	1	1
Γ_2	3	-1	1	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
Γ_3	3	-1	1	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$
Γ_4	6	2	0	0	-1	-1
Γ_5	7	-1	-1	1	0	0
Γ_6	8	0	0	-1	1	1

C.2 Type II: The Δ series

These are the analogues of the dihedral subgroups of $SU(2)$ (i.e., the D series).
Δ_{3n^2}:

	Number of classes	Subgroup of	Some Irreps
$n = 0 \bmod 3$	$8 + \frac{1}{3}n^2$	Full $SU(3)$	9 **1**'s$\frac{1}{3}n^2 - 1$ **3**'s
$n \neq 0 \bmod 3$	$\frac{1}{3}(8 + n^2)$	Full $SU(3)$ and $SU(3)/\mathbb{Z}_3$	3 **1**'s, $\frac{1}{3}(n^2 - 1)$ **3**'s

Δ_{6n^2}:

	Number of classes	Subgroup of	Some Irreps
$n = 0 \bmod 3$	$\frac{1}{6}(24 + 9n + n^2)$	Full $SU(3)$	–
$n \neq 0 \bmod 3$	$\frac{1}{6}(8 + 9n + n^2)$	Full $SU(3)$ and $SU(3)/\mathbb{Z}_3$	2 **1**'s, 1 **2**, $2(n-1)$ **3**'s, $\frac{1}{6}(n^2 - 3n + 2)$ **6**'s

D. Matter content for $\Gamma \subset SU(3)$

Note here that since the $\mathcal{N} = 1$ theory is chiral, the fermion matter matrix need not be symmetric. A graphic representation for some of these theories appear in figures 3, 4 and 5.

$$\begin{array}{ccc} & \text{Fermionic} & \text{Bosonic} \\ \Sigma_{36} & \begin{pmatrix} 2&0&1&1&0&0\\ 0&2&1&1&0&0\\ 1&1&2&0&0&0\\ 1&1&0&2&0&0\\ 0&0&0&0&4&0\\ 0&0&0&0&0&4 \end{pmatrix} & \begin{pmatrix} 2&0&2&2&0&0\\ 0&2&2&2&0&0\\ 2&2&2&0&0&0\\ 2&2&0&2&0&0\\ 0&0&0&0&6&0\\ 0&0&0&0&0&6 \end{pmatrix} \\ & \mathbf{1} \oplus (\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'') & (\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'')^2 \end{array}$$

$$\begin{array}{ccc} \Sigma_{60} & \begin{pmatrix} 1&1&0&0&0\\ 1&2&0&0&1\\ 0&0&1&1&1\\ 0&0&1&2&1\\ 0&1&1&1&2 \end{pmatrix} & \begin{pmatrix} 0&1&1&0&0\\ 1&1&0&1&2\\ 1&0&1&1&2\\ 0&1&1&2&2\\ 0&2&2&2&2 \end{pmatrix} \\ & \mathbf{1} \oplus \mathbf{3} & \mathbf{3} \oplus \mathbf{3}' \end{array}$$

$$\begin{array}{ccc} \Sigma_{72} & \begin{pmatrix} 1&1&0&0&1&0\\ 1&1&0&0&1&0\\ 0&0&1&1&1&0\\ 0&0&1&1&1&0\\ 1&1&1&1&2&0\\ 0&0&0&0&0&4 \end{pmatrix} & \begin{pmatrix} 1&1&0&0&2&0\\ 1&1&0&0&2&0\\ 0&0&1&1&2&0\\ 0&0&1&1&2&0\\ 2&2&2&2&2&0\\ 0&0&0&0&0&6 \end{pmatrix} \\ & \mathbf{1} \oplus (\mathbf{1}' \oplus \mathbf{2}) & (\mathbf{1} \oplus \mathbf{2}) \oplus (\mathbf{1}' \oplus \mathbf{2}) \end{array}$$

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$$
\Sigma_{168} \quad \underset{\mathbf{1} \oplus \mathbf{3}}{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}} \quad \underset{\mathbf{3} \oplus \mathbf{3}'}{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}}
$$

$$
\Sigma_{216} \quad \underset{\mathbf{1} \oplus (\mathbf{1} \oplus \mathbf{2})}{\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}} \quad \underset{(\mathbf{1} \oplus \mathbf{2}) \oplus (\mathbf{1} \oplus \mathbf{2}')}{\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 \end{pmatrix}}
$$

$$
\Sigma_{360} \quad 4\underset{\mathbf{1}^4}{(\delta_{ij})}_{7\times 7} \quad 6\underset{\mathbf{1}^6}{(\delta_{ij})}_{7\times 7}
$$

$$
\Sigma_{36\times 3} \quad \underset{\mathbf{1} \oplus \mathbf{3}}{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{3} \oplus \mathbf{3}'}{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}}
$$

$$
\Sigma_{216\times 3} \quad \underbrace{\begin{pmatrix}
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1
\end{pmatrix}}_{\mathbf{1}\oplus\mathbf{3}} \times
$$

$$
\times \underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&1\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&1&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&1&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&1&1&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&1&0
\end{pmatrix}}_{\mathbf{3}\oplus\mathbf{3}'}
$$

$$
\Sigma_{360\times 3} \underbrace{\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&2\\
0&0&0&0&0&1&1&0&0&1&1&1&0&0&1&0&1
\end{pmatrix}}_{\mathbf{1}\oplus\mathbf{3}} \underbrace{\begin{pmatrix}
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
1&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0\\
0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&1&0&1&0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&0&0&1&0&0&0&0&0&1&1&0&1&1&0&1\\
0&0&1&0&0&0&0&0&0&0&1&1&1&0&1&1&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&1&1\\
0&0&0&0&0&1&1&0&1&1&1&1&0&1&1&0&2\\
0&0&0&0&0&1&1&1&0&1&1&1&1&0&1&2&0
\end{pmatrix}}_{\mathbf{3}\oplus\mathbf{3}'}
$$

Δ_{3n^2} for the following values of n:

$$
n=2 \quad \underbrace{\begin{pmatrix}
1&0&0&2\\
0&1&0&2\\
0&0&1&2\\
2&2&2&4
\end{pmatrix}}_{\mathbf{1}\oplus\mathbf{3}} \quad \underbrace{\begin{pmatrix}
0&0&0&4\\
0&0&0&4\\
0&0&0&4\\
4&4&4&6
\end{pmatrix}}_{\mathbf{3}\oplus\mathbf{3}'}
$$

$$
n=3\quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&3\\
1&1&1&1&1&1&1&1&1&0&1
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&1&0&3\\
1&1&1&1&1&1&1&1&1&3&0
\end{pmatrix}}
$$

$$
n=4\quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&1&0&0&0\\
0&1&0&0&1&0&0&0\\
0&0&1&0&1&0&0&0\\
0&0&0&1&0&1&1&1\\
0&0&0&1&1&2&0&0\\
1&1&1&0&0&1&1&1\\
0&0&0&1&1&0&2&0\\
0&0&0&1&1&0&0&2
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&1&1&0&0\\
0&0&0&0&1&1&0&0\\
0&0&0&0&1&1&0&0\\
0&0&0&0&1&1&2&2\\
1&1&1&1&0&2&1&1\\
1&1&1&1&2&0&1&1\\
0&0&0&2&1&1&2&0\\
0&0&0&2&1&1&0&2
\end{pmatrix}}
$$

$$
n=5\quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&1&1\\
0&0&0&0&1&0&1&1&1&0&0\\
0&0&0&1&0&1&2&0&0&0&0\\
1&1&1&0&0&0&1&0&0&1&1\\
0&0&0&1&0&0&0&2&0&0&1\\
0&0&0&1&0&0&0&0&2&1&0\\
0&0&0&0&1&1&0&0&1&1&0\\
0&0&0&0&1&1&0&1&0&0&1
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&1&0&1&1&1&1\\
0&0&0&1&0&0&1&1&1&1&1\\
1&1&1&1&0&0&2&0&0&1&1\\
1&1&1&0&1&2&0&0&0&1&1\\
0&0&0&1&1&0&0&2&0&0&2\\
0&0&0&1&1&0&0&0&2&2&0\\
0&0&0&1&1&1&1&0&2&0&0\\
0&0&0&1&1&1&1&2&0&0&0
\end{pmatrix}}
$$

$$
n=6\quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&1&2&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&1&2&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&2&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&2
\end{pmatrix}} \times
$$

$$
\times \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&0&2&2&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&2&0&2&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&1&1&2&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&2&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&1&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&0&0&0&1&2
\end{pmatrix}}
$$

Δ_{6n^2} **for the following values of** n

$$n=2\ \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&1&0\\ 0&1&0&0&1\\ 0&0&1&1&1\\ 1&0&1&2&1\\ 0&1&1&1&2 \end{pmatrix}}\ \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&1&1\\ 0&0&0&1&1\\ 0&0&0&2&2\\ 1&1&2&2&2\\ 1&1&2&2&2 \end{pmatrix}}$$

$$n=4\ \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0\\ 0&0&1&1&1&0&0&0&0&0\\ 1&0&1&2&1&0&0&0&0&0\\ 0&1&1&1&2&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&0&0&0&1&0&0&1&1\\ 0&0&0&0&0&1&1&1&1&2 \end{pmatrix}}\ \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&2&2&0&0&0&0&0\\ 1&1&2&2&2&0&0&0&0&0\\ 1&1&2&2&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&2\\ 0&0&0&0&0&0&0&1&1&2\\ 0&0&0&0&0&1&1&0&0&2\\ 0&0&0&0&0&1&1&0&0&2\\ 0&0&0&0&0&2&2&2&2&2 \end{pmatrix}}$$

$$n=6\ \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\ 0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&2&0\\ 0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&2 \end{pmatrix}}\times$$

$$\times\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\ 1&1&2&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&2&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&2&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&2&0\\ 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&2&0\\ 0&0&0&2&2&2&0&0&0&0&0&0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&2&0\\ 0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&0&0&0&2 \end{pmatrix}}$$

$$
n = 8 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}} \times
$$

$$
\times \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}}
$$

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